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Knots and topologically transitive flows on 3-manifolds

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Abstract

Suppose that φ is a nonsingular (fixed point free) flow on a smooth three-dimensional manifold M. Suppose the orbit though a point $p \in M$ is dense in M. Let D be an imbedded disk in M containing p which is transverse to the flow. Suppose that $q \in D$ is a point in the forward orbit of p. Under certain assumptions on M, which include the case $M = S^3$, we prove that if q is sufficiently close to p then the orbit segment from p to q together with a compact segment in D from p to q forms a nontrivial prime knot in M.

Keywords: Dense orbit; Minimal flow; Global cross section; Prime knot; Transitive flow

1. Introduction

We will define some notation from [4] in order to state our main result. Let M be a smooth closed (compact, no boundary) three-dimensional manifold with $H_2(M) = 0$. Our main example of such an M is S^3. Suppose φ is a C^1 nonsingular (fixed point free) topologically transitive (so φ has a dense orbit) flow on M. Let p be any point in the dense orbit. Let D be a compact disk containing p which is transverse to the flow. We call such a disk a transverse disk, and if D is in addition a global cross section, we will call it a global transverse disk. Let $q \in D$ be a point in the forward orbit of p and let $\overrightarrow{pq}$ denote the orbit segment beginning at p and ending q. Let $[pq]$ denote a compact segment in D with endpoints at p and q.

We need some standard terminology from knot theory. For an excellent presentation of many of the important results concerning knots in dynamical systems see the book [3] by Ghrist et al. Let $\Gamma \subset M$ denote a knot. By this we mean that Γ is the image of a continuous injective function from the circle to M. We shall say that Γ is a trivial knot if it bounds a disk. We say that Γ is

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a composite knot if there exists a 2-sphere S in M such that $S \cap \Gamma$ is two points, z and w, and the intersection of each component of $\Gamma - \{z, w\}$ together with a segment in S from z to w is a nontrivial knot. We shall say that Γ is a prime knot if it is neither composite or trivial. When the knot is of class C^1 and

$$\Theta : \Gamma \times \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\} \to M$$

is a C^1 embedding such that, for all $x \in \Gamma$, $\Theta(x, 0, 0) = x$, the concepts of trivial, composite, and prime extend to the torus which is the image of Θ.

Theorem 1. *Under the assumptions above, let* $\Gamma = \overrightarrow{pq} \cup [pq]$ *where* $[pq]$ *is a compact segment in* $D - \overrightarrow{pq} \cap D$ *connecting* p *to* q. *Then* Γ *is a nontrivial prime knot if* q *is close enough to* p. *The result holds in the case* $H_2(M) \neq 0$ *if the flow has not periodic orbits.*

In particular, the dense orbits is proven by Harrison and Pugh in [5] have this knotting property.

Remark. Gutierrez proved a similar result in the excellent paper [4] for a general smooth compact Riemannian manifold M and a minimal flow φ. Our proof follows that in [4] very closely. For the reader who is very familiar with the proof in [4], the following are the most substantial differences between the proofs.

- In [4], the minimality of φ is used to obtain the global transverse disks D_0, D_1, and D_2 from the proof of Theorem 3.1 (in [4]). We instead Ref. [2], where it is proven that any nonsingular C^1 flow on a manifold of dimension greater than 2 has a global transverse disk.
- In [4], the minimality of φ is used to show that any closed (compact, no boundary) manifold N which is a cross section to the flow must be a torus. This is used to show that the foliation constructed in the proof of Theorem 3.1 (in [4]) satisfies criteria (iii) Theorem 2.1 (in [4]). Under our assumption that $H_2(M) = 0$ no such cross section can exist, as $M - N$ would have two components and the flow would cross only from one component to the other violating the assumption of a dense orbit. Under our alternative assumption that the flow has not periodic orbits it is clear that any such cross section must be a torus as the first return map would be fixed point free.

Proof of Theorem 1. We will need the following theorem which appears as Theorem 2.1 in [4].

Theorem 2. *A solid torus* T *contained in* M *is a (nontrivial) prime knot if there exists a transversely orientable bidimensional* C^2 *foliation* $\mathscr{F}$ *on* $\mathscr{V} = \overline{M - T}$ *such that:*

(1) *$\mathscr{F}$ is transversal to $\partial\mathscr{V}$. Moreover, every leaf of $\mathscr{F}$ has nonempty intersection with $\partial\mathscr{V}$.*
(2) *The one-dimensional foliation $\mathscr{F}|_{\partial\mathscr{V}}$ on $\partial\mathscr{V}$ contains a meridian σ as a leaf. Moreover, $\mathscr{F}|_{\partial\mathscr{V}}$ contains no Reeb components.*
(3) *If $\mathscr{F}$ has a compact leaf K, there are finitely many discs $D_1, D_2, \ldots, D_s$ contained in T such that the union of K with $\bigcup_{i=1}^{s} D_i$ is a torus L satisfying $L \cap \partial T = K \cap \partial T = \bigcup_{i=1}^{s} \partial D_i$.*
(4) *Let $B = \{(x, y) \in \mathbb{R}^2 \,|\, 1 \leqslant x^2 + y^2 \leqslant 9$ and $x \leqslant 2\}$ and decompose its boundary ∂B as the union of $B_1 = \{(x, y) \in B \,|\, x^2 + y^2 = 1\}$, $B_2 = \{(x, y) \in B \,|\, x = 2\}$ and $B_3 = \{(x, y) \in B \,|\, x^2 + y^2 = 9\}$.*

There exists an embedding $\lambda : B \times [-1,1] \to \mathscr{V}$ *such that*
(a) $\lambda : (B_1 \cup B_2) \times [-1,1]$ *is precisely the intersection of* $\partial\mathscr{V}$ *with the image* $\mathrm{Im}(\lambda)$ *of* λ.
(b) *The complement of* $\lambda(B_1 \times (-1/2, 1/2))$ *in* $\partial\mathscr{V}$ *is a union of meridians of* $\partial\mathscr{V}$ *which are leaves of* $\mathscr{F}|_{\partial\mathscr{V}}$.
(c) *For all* $p \in B$, *the segments* $\lambda(\{p\} \times [-1,1])$ *are transversal to* $\mathscr{F}$.
(d) *Let* H *be a half straight line of* $\mathbb{R}^2$ *starting at the origin. Then, for all* $z \in [-1,1]$, $\lambda((H \cap B) \times \{z\})$ *is contained in a leaf of* $\mathscr{F}$. *Also, for all* $z \in [-1,-1/2) \cup (1/2,1]$, $\lambda(B \times \{z\})$ *is a plaque of* $\mathscr{F}$.

Proof of Theorem 1 (Conclusion). First, assume that φ is smooth (C^∞). Throughout this proof, if N is a manifold we will use ∂N to denote the boundary of N and int N to denote $N - \partial N$. Let D be a global transverse disk containing p. Such a disk exists by Theorem 2 of [1]. We denote the first return map from D to itself by Υ. Given two submanifolds A and B of D, we denote the first return map (where it is defined) from A to B by $\Upsilon_{[A,B]}$. In the case $A = B$, we use the notation $\Upsilon_{[A,A]} = \Upsilon_{[A]}$.

An M complex, defined below, is a finite cell complex where the "cells" are allowed to be closed manifolds with boundary instead of disks, and these manifolds, called M-cells, are attached along their boundaries.

Definition 1. An n-dimensional *M complex* is a topological space defined as follows.

For each $k = 0, 1, \ldots, n$, let $\{\overline{e^k_\alpha}\}$ be a set of compact k-dimensional manifolds with interiors $\{e^k_\alpha\}$, where α runs over some finite indexing set. The e^k_α are called M-cells, being manifolds which play the role of cells in the definition of a CW complex.

(1) Let $X^0 = \{\overline{e^0_\alpha}\}$ be a discrete set of points.
(2) Inductively define X^k, called the k-skeleton, from X^{k-1} by attaching each $\overline{e^k_\alpha}$ by maps $\psi_\alpha : \partial\overline{e^k_\alpha} \to X^{k-1}$. That is, X^k is the identification space of $X^{k-1} \coprod_\alpha \overline{e^k_\alpha}$ under $x \sim \psi_\alpha(x)$ for $x \in \partial\overline{e^k_\alpha}$.

Following the notational conventions in [6] for CW complexes, if C denotes the set of cells and attaching maps then $|C| = \bigcup_n X^n$ denotes the resulting topological space. Hence an M-cell e^k_α will be considered as a element of C and a submanifold of $|C|$.

The following is our definition of M-cellwise continuous.

Definition 2. Suppose that C_d and C_r are M complexes and $h : |C_\mathrm{d}| \to |C_\mathrm{r}|$ is a (not necessarily continuous) map. (The notation is chosen because C_d is the cell complex on the domain of h and C_r is the cell complex on the range of h.) If h restricted to any M-cell of C_d is continuous and the image of any M-cell of C_d under h is a M-cell of C_r then we say h is M-cellwise continuous. For us, h will be a bijection.

For $x \in D$, let $\Delta(x) = \max\{n > 0 \mid \Upsilon^n(x) \in \mathrm{int}\, D\} - \min\{n \leqslant 0 \mid \Upsilon^n(x) \in \mathrm{int}\, D\}$. It is proven in [1] that there exists a arbitrarily small perturbation of D such that there exists two M-cell complexes, C_d and C_r on D satisfying the following.

(1) Υ is M-cellwise continuous for the cell complexes C_d and C_r and, hence, the restriction of Υ to any M-cell of C_d is a homeomorphism.

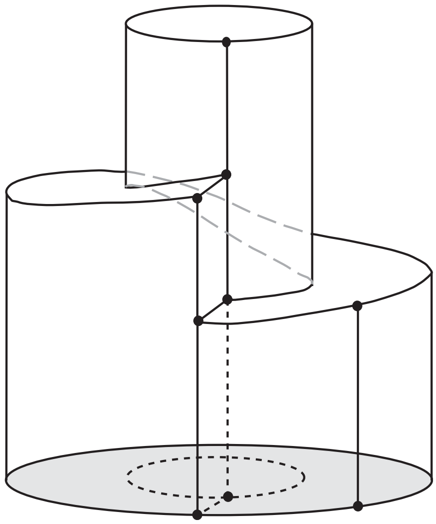

Fig. 1. The tower T for a flow on a manifold that is homotopic to S^3.

(2) A point $x \in D$ is in a 2-cell of C_{d} if and only if $\Delta(x) = 1$.
(3) A point $x \in D$ is in a 1-cell of C_{d} if and only if $\Delta(x) = 2$.
(4) A point $x \in D$ is in a 0-cell of C_{d} if and only if $\Delta(x) = 3$.

Moreover, it is clear that the disk D can be chosen as a C^∞ smooth submanifold and hence the 1-cells of C_{d} are C^∞ curves.

There is a natural way to represent the flow on M as the union of flowboxes with their bases all on D. For $x \in D$, let $\tau(x) = \min\{t > 0 \mid \varphi(t,x) \in D\}$ denote the first return time for x. For $x \in M$, let $\tau_0(x) = \max\{t \leqslant 0 \mid \varphi(t,x) \in D\}$. Define the map $F : M \to \mathbb{R} \times D$ by

$$F(x) = (-\tau_0(x), \varphi(\tau_0(x), x)).$$

Note that $\varphi \circ F$ is the identity map on M. Then we define the "tower" associated with D to be the set

$$T = \overline{\mathrm{Im}(F)}.$$

Note that $\varphi : T \to M$ is a identification map. An example of the tower for a flow on a manifold that is homotopic to S^3 is shown in Fig. 1. This example is investigated in [1]. Note that the pull back of the flow on M by φ is a semi-flow on T where all orbits move strictly "up", that is, in the positive $\mathbb{R}$ direction.

There exists the natural projection $\pi : T \to D$ defined by

$$\pi(t,x) = x.$$

Then for $x \in D$, $\varphi(\pi^{-1}(x))$ is the orbit segment from x to $\Upsilon(x)$. For each n-cell A of C_{d}, $\pi^{-1}(A)$ is a flowbox and we call $\varphi(\pi^{-1}(A) \cap \operatorname{int} T)$ the vertical $(n+1)$-cell over A. For each 2-cell A of C_{d}, let x_A be the second point at which the forward orbit through p intersects A and let $y_A = \Upsilon(x_A)$. Note that $\varphi(\pi^{-1}(x_A)) = \overrightarrow{x_A y_A}$.

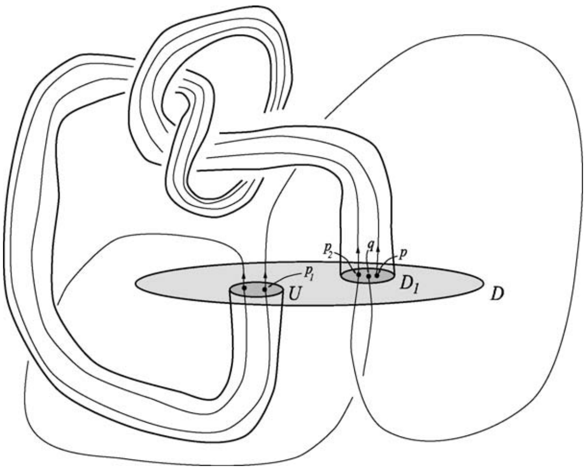

Fig. 2. The flowbox W imbedded in M. The intersections between $\overrightarrow{pq}$ and D other than p, p_1, p_2, and q are not shown for simplicity.

In the following, we define a two-dimensional foliation of M that is induced by the definitions above. The foliation will be transverse to the flow and we will define the foliation M in pieces. First, define the foliation on D that has D as a leaf.

For each 1-cell J of C_{d}, we can construct a one-dimensional smooth foliation on $\pi^{-1}(J)$ that has the bottom (entry transversal, J) and top (exit transversal) as leaves. Define a smooth foliation on $\varphi(\pi^{-1}(J))$ as the push forward by φ of the foliation on $\pi^{-1}(J)$. If J and K are 1-cells of C_{d}, then $\bar{J} \cap \bar{K}$ is either one or two points. Hence the closure of the vertical cells over J and K intersect along $\varphi(\pi^{-1}(F(\bar{J} \cap \bar{K})))$, which is either one or two obit segments. Hence, this defines a smooth foliation on the union of D and all vertical 1-cells and 2-cells.

For each 2-cell $A \in C_{\mathrm{d}}$, let $R : (\bar{A} - \{x_A\}) \to \partial\bar{A}$ be a smooth retraction which is a submersion. Define $P : \pi^{-1}(\bar{A} - \{x_A\}) \to \pi^{-1}(\partial\bar{A})$ by $P(t,x) = (t, R(x))$. Then the foliation on $\mathrm{Im}(P)$ pulls back by P to a smooth foliation on $\pi^{-1}(\bar{A} - \{x_A\})$ which is transverse to the flow. We can extend this foliation to $\pi^{-1}(\bar{A} - \{x_A\}) \cup \{(0,x_A),(\tau(x_A),x_A)\}$ by defining the top and bottom of the flowbox $\pi^{-1}(\bar{A})$ to be leaves of the foliation. This foliation pulls back, via F, to a foliation on $\varphi(\pi^{-1}((\bar{A} - \{x_A\}) \cup \{(0,x_A),(\tau(x_A),x_A)\}))$. The foliations on the closures of vertical 3-cells are compatible as the only place where any two closures of vertical 3-cells meet is along a vertical 1 or 2-cell or in D, and each component of D intersected with the closure of a vertical 3-cell is a leaf of the foliation on the closure of the 3-cell.

Let $\mathscr{F}$ be the foliation obtained above on the compliment of $\bigcup_{A \in C_{\mathrm{d}}} \overrightarrow{x_A y_A}$.

Let $p_1 \in D$ be a point in the forward orbit of p such that the orbit segment $\overrightarrow{p p_1}$ intersects every 2-cell of C_{d} at least 3 times. Define D_1 to be an open disk with $p \in D_1 \subset D$ with $D_1 \cap \overrightarrow{p p_1} = p$. (See Fig. 2.) Suppose, furthermore, that D_1 is chosen small enough that D_1 is contained in the 2-cell of C_{d} which contains p. Let U be a disk in D containing p_1. If D_1 and U are small enough, there is a well defined first intersection map $\Upsilon_{[D_1,U]}$ with $\Upsilon_{[D_1,U]}(p) = p_1$. Moreover, U can be chosen so that $\Upsilon_{[D_1,U]}$ is a homeomorphism. Let $p_2 = \Upsilon_{[D_1]}(p)$ and $q = \Upsilon_{[D_1]}(p_2)$. That is, p_2 is the first return of p to D_1 and q is the second return of p to D_1. Let $[pq]$ be a compact segment in D_1

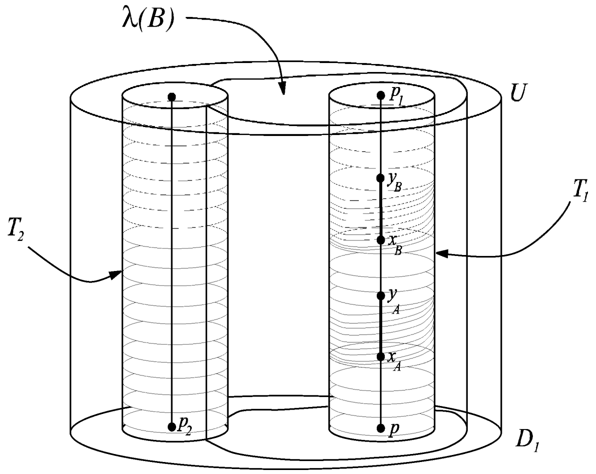

Fig. 3. The imbedding $\lambda(B)$ inside the flowbox W.

with $[pq] \cap \overrightarrow{pq} = \{p,q\}$. Let W be the flowbox with entrance transversal D_1 and exit transversal U. This is shown in Fig. 2. Observe that $\overrightarrow{pq} \cap W$ consists of the point q and the two orbit segments $\overrightarrow{pp_1}$ and $\overrightarrow{p_2\, \Upsilon_{[D_1,U]}(p_2)}$. Also observe that $\overrightarrow{x_A y_A} \subset \overrightarrow{pp_1}$ for all 2-cells $A \in C_{\mathrm{d}}$.

Let $\Gamma = \overrightarrow{pq} \cup [pq]$. Let T be a solid torus which is a tubular neighborhood of Γ with ∂T transverse to the foliation $\mathscr{F}$. Let $\mathscr{V} = \overline{M - T}$. We can assume that the leaves of $\mathscr{F}$ intersect T in disks in a neighborhood of $[pq]$ since the foliation is trivial in a neighborhood of $[pq]$. If T is chosen small enough then $T \cap W$ is two disjoint closed cylinders, one of which contains all of the $\overrightarrow{x_A y_A}$.

It is clear that items (1) and (2) of Theorem 2 hold for the foliation $\mathscr{F}$ from the definitions above. To prove that item (3) holds, suppose that $\mathscr{F}$ has a compact leaf K. Then $K \cap \partial T$ must be compact and hence the union of finitely many close curves. Each of these must be a meridian of ∂T since $\mathscr{F}|_{\partial T}$ contains a meridian as a leaf by criteria (1). Hence there are finitely many disks $D_1, \ldots, D_s$ such that $N = K \bigcup_i D_i$ is a closed manifold transverse to the flow. If $H_2(M) = 0$, $M - N$ has two components, and the vector field on N points from one component into the other. This violates the assumption that the flow has a dense orbit. Under the alternate assumption that $H_2(M) \neq 0$ but φ has no periodic orbits, the first return map to N must be fixed point free and hence N must be a torus.

To prove that item (4) holds, recall that T is chosen small enough that $T \cap W$ is the union of two cylinders. One of these is a tubular neighborhood of $\overrightarrow{pp_1}$, which we call T_1, and the other is a neighborhood of $\overrightarrow{p_2 q}$, which we call T_2. Also recall that $\overrightarrow{x_A y_a} \subset \overrightarrow{pp_1}$ for all two cells $A \in C_{\mathrm{d}}$. Hence, all of the leaves of $\mathscr{F}|_{\partial T}$ that are not meridians are in contained in ∂T_1. Then it is clear that the imbedding λ required in item (4) exists with $\lambda(B_1) = T_1$ and $\lambda(B_2) = T_2$ as shown in Fig. 3.

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